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# A complete second-order theory for the unsteady flow about an airfoil due to a periodic gust

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In this paper we develop a uniformly valid, second-order theory for calculating the unsteady incompressible flow that occurs when an airfoil is subjected to a convected sinusoidal gust. Explicit formulae for the airfoil response functions (i.e. fluctuating lift) are given. The theory accounts for the effect of the distortion of the gust by the steady-state potential flow around the airfoil, and this effect is found to have an important influence on the response functions. A number of results relevant to the general theory of the scattering of vorticity waves by solid objects are also presented.

## 1. Introduction

The theory of unsteady flows around stationary airfoils has numerous important technological applications. It is, for example, a fundamental ingredient in any calculation of the unsteady blade forces that are the source of such a wide variety of undesirable effects in turbomachinery.

This theory has been developed primarily in the linearized approximation, wherein the unsteady flow is decoupled from the steady-state aerodynamics (Küssner 1940; Sears 1941; and others). In fact, at this level of approximation, the unsteady lift on an airfoil is the same as that on a flat plate with zero thickness and angle of attack. Recently, Horlock (1968) has (by means of a heuristic approach) partially accounted for some of the coupling between the angle of attack of the airfoil and the unsteady flow. Similar ideas have been used by Naumann & Yeh (1972) to account for camber. These theories suffer from the drawback of including some of the coupling effects while not including others.

In order to formulate correctly the problem of a non-uniform flow around a stationary airfoil, it is necessary to consider two small parameters. One of these, termed $\epsilon$, is the amplitude of the unsteady incident disturbance; the other, termed $\alpha$, is a measure of the angle of attack, camber or thickness of the airfoil (i.e. the steady-flow disturbance caused by the airfoil). The linear theory accounts for the $O(\epsilon)$ effects, while the coupling terms are $O(\alpha\epsilon)$. We can ensure that all such terms are accounted for only by developing a systematic expansion. Such an approach is taken in this paper. Although this is the first time that this has been done for the gust problem, several authors, beginning with Van Dyke (1954), have developed second-order expansions for the problem of an oscillating airfoil in an irrotational flow.

In order to concentrate on the coupling effect, we suppose that $\epsilon = o(\alpha)$. Physically, this amounts to requiring that the amplitude of the gust be much smaller than the steady-state disturbance.

One of the new effects that is included in this approach is the distortion of the oncoming gust by the steady-state potential flow field about the airfoil. This distortion acts to cause significant variations in the wavelength of the incident vorticity wave while also causing variations in both the amplitude and phase of its associated velocity field. The details of this nonlinear dispersion process will be discussed subsequently, and a number of results relevant to the general theory of vorticity-wave scattering will be given. Moreover, it will be shown that this phenomenon has such an important effect on the fluctuating lift that it introduces a term that exactly cancels those occurring in Horlock's theory. As a result, our formula for the fluctuating lift is quite different from the one obtained by Horlock. In fact, the distortion effect causes the fluctuating lift to depend on the wavenumber of the gust in the direction perpendicular to the plane of the airfoil. No previous theory exhibits such a dependence.

In § 2 we formulate the general problem of a two-dimensional airfoil in an incompressible flow subject to a small amplitude gust and integrate the vorticity equation that governs this process. In § 3 the results are restricted to the case where the steady-flow disturbance caused by the airfoil is small, and a formal asymptotic expansion of the unsteady solution is constructed. However, this expansion turns out to be non-uniformly valid at large distances from the airfoil, and it is necessary to construct an appropriate 'outer expansion' for this region. The matching of these two expansions provides a boundary condition (at infinity) on the homogeneous solution to the 'inner' problem. (The corresponding particular solution is given in § 3.1.) The boundary conditions on the surface of the body are deduced in appendix C, and in § 3.1 we present the homogeneous solution that satisfies these conditions as well as the one at infinity. We then show that this solution is non-uniformly valid at the leading and trailing edges and use the method of strained co-ordinates (Lighthill 1951) to make it uniformly valid at these points. The physical implications of the solution are discussed in § 3.2, while in § 4 we derive a formula for the fluctuating lift on an airfoil of arbitrary shape and thickness distribution.

In contrast with the case of linearized steady flow (or for that matter the case of a second-order unsteady flow around an oscillating airfoil), the effects of thickness, camber and angle of attack cannot simply be superposed, primarily because the distortion of the gust imparts a nonlinear character to the problem. However, it is shown in § 4.2 that the results for zero-thickness airfoils are much simpler than those for airfoils with thickness, and an explicit formula (in terms of Bessel functions) is obtained for the flat-plate airfoil at an angle of attack to the mean flow. Finally, the physical implications of the flat-plate solutions are discussed in § 4.3.

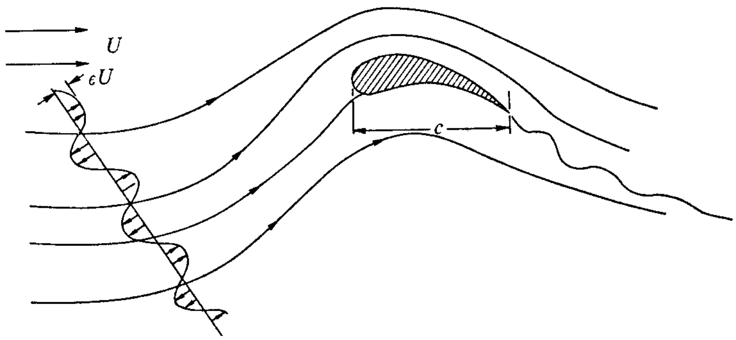


FIGURE 1. Gust approaching airfoil.

## 2. Formulation

Consider a two-dimensional airfoil with chord length $c$ placed in a uniform stream having mean velocity $U$ at large distances from the airfoil (figure 1). As in the Sears problem, we suppose that a frozen convected sinusoidal gust whose amplitude $\epsilon U$ is much less than the free-stream velocity $U$ is imposed on the flow far upstream from the airfoil. We further suppose that the flow is two-dimensional, incompressible and inviscid and that body forces can be neglected. Then Euler's equations become

$$\nabla . \mathbf{V} = 0 \tag{2.1}$$

and

$$(\partial/\partial t + \mathbf{V} . \nabla) \mathbf{V} = -\nabla P, \tag{2.2}$$

where all lengths have been non-dimensionalized by $\frac{1}{2}c$, the time $t$ by $c/2U$, the velocity $\mathbf{V}$ by $U$, and the pressure $P$ by $\rho U^2$.

Since the steady flow is assumed to be inviscid and irrotational, the solution must be of the form

$$\mathbf{V} = \mathbf{v}(\mathbf{x}) + \epsilon \mathbf{u}(\mathbf{x}, t) + \ldots, \tag{2.3}$$

$$P = p_s(\mathbf{x}) + \epsilon p'(\mathbf{x}, t) + \ldots, \tag{2.4}$$

where the steady velocity $\mathbf{v}(\mathbf{x})$ satisfies the conditions

$$\nabla . \mathbf{v} = \nabla \times \mathbf{v} = 0, \tag{2.5}$$

and $\mathbf{u}$ and $p'$ are of order unity. Taking the curl of (2.2), using (2.1) to introduce a stream function $\psi$ for the unsteady velocity, and neglecting terms of order $\epsilon^2$ (§ 1, end of third paragraph) yields

$$(\partial/\partial t + \mathbf{v} . \nabla) \Omega = 0, \tag{2.6}$$

where $\Omega$, which denotes the negative of the vorticity, is given by

$$\Omega = \nabla^2 \psi = \partial u_1/\partial x_2 - \partial u_2/\partial x_1; \tag{2.7}$$

$\psi$, which determines the unsteady velocity field, is given by

$$\mathbf{u} = \{u_1, u_2\} = \{\partial\psi/\partial x_2, -\partial\psi/\partial x_1\}; \tag{2.8}$$

and $\mathbf{x} = \{x_1, x_2\}$ are Cartesian co-ordinates with $x_1$ aligned in the direction of the upstream mean velocity $U$.

Equation (2.6) can readily be solved for $\Omega$ by introducing the steady-flow potential and stream function $\Phi$ and $\Psi$, respectively, to obtain the equation

$$(\partial/\partial t+|\mathbf{v}|^2\,\partial/\partial\Phi)\,\Omega = 0, \tag{2.9}$$

whose general solution is

$$\Omega = f\left(\int_{-\infty}^{\Phi}\left(\frac{1}{|\mathbf{v}|^2}-1\right)d\Phi+\Phi-t,\Psi\right), \tag{2.10}$$

where $f$ is an arbitrary function of its arguments. Far upstream (where the 'scattered' field produced by the airfoil decays), this solution must approach the vorticity distribution of the imposed gust, which was required to be periodic in time. But this can occur only if the function $f$ is chosen such that

$$\Omega = g(\Psi)\exp\left\{ik_1\left[\int_{-\infty}^{\Phi}\left(\frac{1}{|\mathbf{v}|^2}-1\right)d\Phi+\Phi-\Phi_0-t\right]\right\}, \tag{2.11}$$

where $k_1$ is the non-dimensional frequency, $\Phi_0 = \lim_{x_1\to-\infty}[\Phi(x_1,x_2)-x_1] = \text{constant}$ (with $x_2$ finite) and $g$ is essentially an arbitrary function of $\Psi$. Since the problem is linear, we can obtain a solution that corresponds to any choice of the function $g$ by superposing solutions associated with the individual harmonic components:

$$g(\Psi) = -i\,|k|\exp\left[ik_2(\Psi-E_0)\right], \tag{2.12}$$

where $k_2$ is the wavenumber of this component and $E_0$ is a constant. The normalization $-i\,|k|$, where

$$k = k_1+ik_2, \tag{2.13}$$

is chosen simply as a matter of convenience. Then without loss of generality we can take

$$\Omega = -i\,|k|\exp\left(i\left\{k_1\left[\int_{-\infty}^{\Phi}\left(\frac{1}{|\mathbf{v}|^2}-1\right)d\Phi+\Phi-\Phi_0-t\right]+k_2(\Psi-E_0)\right\}\right). \tag{2.14}$$

This equation determines the vorticity field everywhere around the airfoil, including the region far upstream.

In the purely linear problem (Sears 1941) the vorticity is given by

$$\Omega = -i\,|k|\exp\{i[k_1(x_1-t)+k_2x_2]\}. \tag{2.15}$$

Far upstream from the airfoil, where $\Phi-\Phi_0 \sim x_1$, $|\mathbf{v}| \sim 1$ and $\Psi-E_0-x_2$ must behave like $\ln|\mathbf{x}|$ for any lifting airfoil (Milne-Thomson 1962, p. 194), the vorticity wave (2.14) will not reduce precisely to (2.15) no matter how weak the lift of the airfoil may be (i.e. no matter how small the coefficient of $\ln|\mathbf{x}|$).

Since the local wavelength of the vorticity wave (2.14) is precisely $2\pi|\mathbf{v}|/k_1$, it is clear that this quantity is strongly affected by the steady velocity field and will not remain constant, as it does for the completely linearized wave (2.15). In fact, the wavelength will be longer on streamlines that pass over the top of the airfoil and shorter on those that pass below. Equation (2.14) also shows that the amplitude of the vorticity wave is conserved.

The remaining boundary conditions are that

$$\mathbf{u}.\hat{\mathbf{n}} = 0 \quad \text{for } \mathbf{x} \text{ on } S \tag{2.16}$$

(where $\hat{\mathbf{n}}$ is the unit normal to the surface $S$ of the airfoil) and that the pressure and normal velocity be continuous across any inviscid vortex wake that forms downstream of the airfoil. Finally we require that the airfoil have a sharp trailing edge at which the Kutta condition is always satisfied. Then the problem amounts to solving the Poisson equation determined by (2.7) and (2.14) subject to these boundary conditions.

## 3. Linearized problem

In order to obtain a relatively simple closed-form solution, the analysis is restricted to the case of a thin airfoil with a small angle of attack and camber. Thus, let $\alpha$ denote a small parameter that is characteristic of the steady-flow disturbance caused by the airfoil. Then the associated velocity field must be of the form

$$\mathbf{v}(\mathbf{x}) = \hat{\mathbf{i}} + \alpha \mathbf{v}^{(1)}(\mathbf{x}), \tag{3.1}$$

where $\hat{\mathbf{i}}$ is a unit vector in the $x_1$ direction and $\mathbf{v}^{(1)}$ is of order unity.

### 3.1. *Solution*

*Inner expansion.* Instead of working with (2.14) directly, it is more convenient to return to the unintegrated equations (2.6)–(2.8). The structure of these equations suggests that we seek solutions of the form

$$\mathbf{u} = \exp(-ik_1 t)\,[\mathbf{u}^{(0)}(\mathbf{x}) + \alpha \mathbf{u}^{(1)}(\mathbf{x}) + \ldots], \tag{3.2}$$

$$p' = \exp(-ik_1 t)\,[p^{(0)}(\mathbf{x}) + \alpha p^{(1)}(\mathbf{x}) + \ldots], \tag{3.3}$$

$$\psi = \exp(-ik_1 t)\,[\psi^{(0)}(\mathbf{x}) + \alpha \psi^{(1)}(\mathbf{x}) + \ldots]. \tag{3.4}$$

Then substituting (3.4) into (2.6) and (2.7) and equating like powers of $\alpha$ shows that

$$(-ik_1 + \partial/\partial x_1)\,\nabla^2 \psi^{(0)} = 0 \tag{3.5}$$

and

$$(-ik_1 + \partial/\partial x_1)\,\nabla^2 \psi^{(1)} = -\mathbf{v}^{(1)}.\nabla[\nabla^2 \psi^{(0)}], \tag{3.6}$$

where, of course, both $\mathbf{u}^{(0)}$ and $\psi^{(0)}$, and $\mathbf{u}^{(1)}$ and $\psi^{(1)}$ arerelated by equations of the type (2.8).

Equation (3.5) (which corresponds to the usual unsteady, linearized, thin-airfoil theory of Sears) can be integrated at once to obtain $\nabla^2\psi^{(0)} = -i\,|\mathbf{k}|\exp(i\mathbf{k}.\mathbf{x})$, where $\mathbf{k}$ denotes the vector $\{k_1, k_2\}$ and the normalization has been chosen to be compatible with (2.15). Substituting this into (3.6) yields

$$(-ik_1 + \partial/\partial x_1)\,\nabla^2 \psi^{(1)} = -|k|\,\mathbf{k}.\mathbf{v}^{(1)} e^{i\mathbf{k}.\mathbf{x}}. \tag{3.7}$$

In the remainder of this paper we assume (in order to simplify the equations) that $k_1$ and $k_2$ are both positive. No generality is lost by assuming that one of these, say $k_1$, is always positive; but it is then necessary to consider the case where $k_2$ is negative. The results for this case will simply be stated at the end. Their derivation, of course, is nearly identical to the one given below.

In order to solve (3.7) it is convenient to introduce the analytic function $\zeta^{(1)}(z) \equiv v_1^{(1)} - iv_2^{(1)}$ of the complex variable $z$ together with (2.13) to obtain (since $\mathbf{k}.\mathbf{v}^{(1)} = \mathrm{Re}\,(k\zeta^{(1)})$)

$$4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left(-ik_1 + \frac{\partial}{\partial x_1}\right)\psi^{(1)} = -\frac{|k|}{2}\,(k\zeta^{(1)} + \overline{k\zeta^{(1)}})\exp[\tfrac{1}{2}i(k\bar{z} + \bar{k}z)], \tag{3.8}$$

where overbars denote complex conjugates and $\partial/\partial z$ denotes the partial derivative taken with respect to $z$ while $\bar{z}$ is held fixed. Since the right side of this equation is the sum of two terms each of which is the product of a function of $z$ and a function of $\bar{z}$, it can be integrated first with respect to $z$, then with respect to $\bar{z}$, and finally with respect to $x_1$ to obtain

$$\psi^{(1)} = -\frac{1}{|k|}\left[\frac{k}{2}\mathscr{I}_+ - \overline{\frac{k}{2}\mathscr{I}_-} - e^{i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{Re}\,(kW^{(1)}(z))\right] + f(x_2)\,e^{ik_1x_1} + \tilde{F}(z) + \tilde{G}(\bar{z}), \quad (3.9)$$

where $f(x_2)$ is an arbitrary function of $x_2$; $\tilde{F}$ and $\tilde{G}$ are arbitrary analytic functions of $z$ and $\bar{z}$, respectively; $W^{(1)} = \Phi^{(1)} + i\Psi^{(1)}$ is the complex potential associated with $\zeta^{(1)}$ in the usual way by

$$dW^{(1)}/dz = \zeta^{(1)}; \quad (3.10)$$

and

$$\mathscr{I}_\pm(\mathbf{k}, \mathbf{x}) = \pm\, e^{\pm\frac{1}{2}ik\bar{z}}\,\mathscr{K}_\pm(z), \quad (3.11)$$

where

$$\mathscr{K}_\pm(z) \equiv \int_{\mp\infty}^{z} \zeta^{(1)}(z)\, e^{\pm\frac{1}{2}i\bar{k}z}\, dz \quad (3.12)$$

are, of course, analytic functions of $z$. But it follows from the fact that $\zeta^{(1)}(z)$ behaves like $i\Gamma/z$ for large $z$, where $\Gamma$ is a constant, that these functions are actually multivalued. We therefore choose the branch cut of $\mathscr{K}_+$ to lie along the positive real axis and that of $\mathscr{K}_-$ to lie along the negative real axis.

It is now easy to show by differentiating (3.9) that the $O(\alpha\epsilon)$ velocity $\{u_1^{(1)}, u_2^{(1)}\}$ can be expressed as the sum

$$\{u_1^{(1)}, u_2^{(1)}\} = \{u_1^p + u_1^h, u_2^p + u_2^h\} \quad (3.13)$$

of a *homogeneous* solution and a *particular* solution. The homogeneous solution is

$$\left.\begin{aligned} u_1^h &\equiv f'(x_2)\exp(ik_1x_1) + F(z) + G(\bar{z}), \\ u_2^h &\equiv -ik_1 f(x_2)\exp(ik_1x_1) + i[F(z) - G(\bar{z})], \end{aligned}\right\} \quad (3.14)$$

where the arbitrary function $f$ of $x_2$ and the arbitrary analytic functions $F$ and $G$ of $z$ and $\bar{z}$, respectively, will subsequently be determined such that $\mathbf{u}_1^{(1)}$ satisfies the linearized boundary conditions deduced in appendix C. The particular solution is

$$\left.\begin{aligned} u_1^p(x_1, x_2) &\equiv -|k|^{-1}\{J_+ - \bar{J}_- - ik_2 e^{i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{Re}\,\{k[W^{(1)}(z) - W_0]\}), \\ u_2^p(x_1, x_2) &\equiv |k|^{-1}(J_+ + \bar{J}_- - k_1 e^{i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{Re}\,\{k[W^{(1)}(z) - W_0]\}), \end{aligned}\right\} \quad (3.15)$$

where

$$J_\pm(\mathbf{k}, \mathbf{x}) \equiv \tfrac{1}{4}k^2 e^{\pm\frac{1}{2}ik\bar{z}}[\mathscr{K}_\pm(z) - D_\pm \overline{\mathscr{K}_\pm(z)}] \quad (3.16)$$

and $W_0$ is a complex constant that we shall determine subsequently. $D_\pm$ are constants, and are set equal to

$$D_\pm = \int_{-1}^{1} \Delta\zeta^{(1)}(x_1)\exp(\pm\tfrac{1}{2}i\bar{k}x_1)\,dx_1 \Big/ \int_{-1}^{1} \overline{\Delta\zeta^{(1)}(x_1)}\exp(\mp\tfrac{1}{2}ikx_1)\,dx_1, \quad (3.17)$$

where for any function $f(x_1, x_2)$ the notation $\Delta f(x_1)$ is used to denote the jump in $f$ across the real axis at the point $x_1$. This choice of $D_\pm$ is made to ensure that (see figure 2)

$$\Delta u_1^{(1)}(x_1) = \Delta u_1^h(x_1), \quad \Delta u_2^{(1)}(x_1) = \Delta u_2^h(x_1) \quad \text{for} \quad x_1 < -1 \quad (3.18)$$

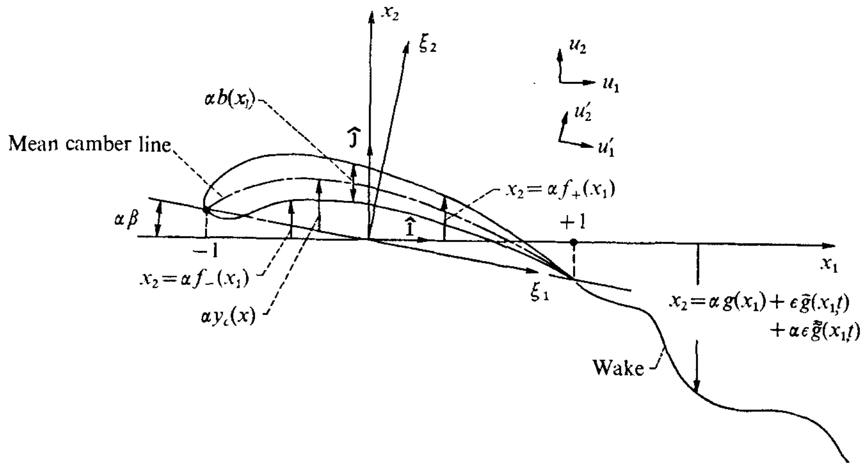


FIGURE 2. Airfoil and wake geometry.

(as can be seen by using the fact that $\Delta\zeta^{(1)}(x_1) = \Delta W^{(1)}(x_1) = 0$ for the region $x_1 < -1$ ahead of the airfoil and inserting (3.12), (3.16) and (3.17) in (3.13) and (3.14)), and that

$$\left.\begin{aligned} \Delta u_1^{(1)}(x_1) &= ik_2|k|^{-1}\exp(ik_1x_1)\,\mathrm{Re}\{k\,\Delta W^{(1)}(1)\} + \Delta u_1^h(x_1) \\ \Delta u_2^{(1)}(x_1) &= -ik_1|k|^{-1}\exp(ik_1x_1)\,\mathrm{Re}\{k\,\Delta W^{(1)}(1)\} + \Delta u_2^h(x_1) \end{aligned}\right\} \text{ for } x_1 \geqslant 1 \tag{3.19}$$

(as can be seen by using the preceding equations and the fact that $\Delta\zeta^{(1)}(x_1) = 0$ and $\Delta W^{(1)}(x_1)$ is constant in the region $x_1 > 1$ behind the airfoil).

*Outer expansion: boundary conditions at infinity.* The solution $\mathbf{u}^{(0)}$ to the completely linearized problem certainly remains bounded as $z \to \infty$ (appendix B). But we now show that the $O(\alpha\epsilon)$ solution $\mathbf{u}^{(1)}$ [given by (3.13)–(3.16)] becomes infinite there. Hence (3.2) is non-uniformly valid at infinity, and it is necessary to construct an 'outer' expansion for this region. Before doing this we must prove that $\mathbf{u}^{(1)}$ becomes infinite as $z \to \infty$. To this end, recall that, as long as the airfoil has lift, $W^{(1)}(z) \sim i\Gamma \ln z$ as $z \to \infty$. Therefore $\mathbf{u}^{(1)}$ contains a term that becomes infinite like $|k|^{-1}\exp(i\mathbf{k}.\mathbf{x})\,\mathrm{Re}\{ik\Gamma\ln(z)\}$ as $z \to \infty$. Inserting the results of appendix A into (3.16) shows that $J_\pm$ are $O(z^{-1})$ as $z \to \infty$, and it is not difficult to see that it is impossible to choose the functions $f$, $F$ and $G$ in (3.14) to cancel the infinite term in $\mathbf{u}^{(1)}$ for all values of $z$. Hence $\mathbf{u}^{(1)}$ must certainly become infinite as $z \to \infty$.

In order to construct the outer expansion, notice that it follows from the theory of steady-state, two-dimensional, potential flows (Milne-Thomson 1962, p. 194) that as $z \to \infty$

$$\Phi + i\Psi - (\Phi_0 + iE_0) = z + \alpha[i\Gamma\ln z + (a+ib)z^{-1} + i(e-e_0) + O(z^{-2})], \tag{3.20}$$

where $a$, $b$ and $e$ are real constants and we have put $E_0 = \alpha e_0$ in order to obtain agreement with (2.15). Then, since $|d(\Phi+i\Psi)/dz|^2 = |\mathbf{v}|^2$, this result can be inserted into (2.14) to obtain (once the integrations have been carried out and (2.7) inserted into the result)

$$\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} = -i|k|\exp\{i[\mathbf{k}.\mathbf{x} - \alpha\,\mathrm{Re}\{k(W^{(1)}(z) - W_0)\} - k_1t]\} + O\left(\alpha^2, \frac{\alpha}{|z|^2}\right)$$
$$\text{as} \quad z \to \infty, \alpha \to 0, \quad 0 < \arg z < 2\pi. \tag{3.21}$$

where we have defined† the complex constant $W_0$ to be

$$W_0 = \lim \Phi^{(1)}(x_1, x_2) + ie_0 \quad \text{as} \quad x_1 \to \infty \quad \text{with } x_2 \text{ finite.} \tag{3.22}$$

Equations (2.8) and (3.21) determine the velocity field at large distances from the body. It is easy to verify that they are satisfied to within an error $O(\alpha^2, \alpha/|z|^2)$ by

$$\left.\begin{aligned} u_1^{\text{out}} &= -\frac{1}{|k|}\exp\{i[\mathbf{k}.\mathbf{x} - \alpha \operatorname{Re}\{k(W^{(1)}(z) - W_0)\} - k_1 t]\}\left(k_2 + \alpha\Gamma \operatorname{Re}\frac{k^2}{\bar{k}z}\right) \\ &\qquad + [\mathscr{F}(z) + \mathscr{G}(\bar{z})]\exp(-ik_1 t), \\ u_2^{\text{out}} &= \frac{1}{|k|}\exp\{i[\mathbf{k}.\mathbf{x} - \alpha \operatorname{Re}\{k(W^{(1)}(z) - W_0)\} - k_1 t]\}\left(k_1 - \alpha\Gamma \operatorname{Im}\frac{k^2}{\bar{k}z}\right) \\ &\qquad + i[\mathscr{F}(z) - \mathscr{G}(\bar{z})]\exp(-ik_1 t), \end{aligned}\right\} \tag{3.23}$$

where $\mathscr{F}$ and $\mathscr{G}$ are arbitrary analytic functions of their arguments. Substituting this result into the momentum equation (2) (with only terms through $O(\alpha\epsilon)$ retained) shows that the unsteady pressure can remain uniformly bounded as $z \to \infty$ only if there is a constant $M$ such that

$$\mathscr{F} = M/z + o(z^{-1}) \quad \text{and} \quad \mathscr{G} = -M/\bar{z} + o(\bar{z}^{-1}).$$

The pressure fluctuation will then behave like $M\exp(-ik_1 t)\tan^{-1}(y/x)$. But since this function is discontinuous along some curve in the $x$, $y$ plane, we can satisfy the requirement that the pressure be continuous only by putting $M = 0$, which implies that $\mathscr{F} = o(z^{-1})$ and $\mathscr{G} = o(\bar{z}^{-1})$ as $|z| \to \infty$.

By using the results of appendix A to expand the inner solution (3.2) (for large $z$) with $\mathbf{u}^{(0)}$ given by (B 1) and $\mathbf{u}^{(1)}$ given by (3.13)–(3.16), it can now be shown that the inner and outer expansions of the velocity can be matched in some intermediate domain only if $u_1^h$, $u_2^h \to 0$ as $|z| \to \infty$. However, it follows from the momentum equation (2.2) that the inner and outer expansions for the pressure fluctuation will match only if the more severe requirement

$$u_1^h = -\frac{\Gamma}{2|k|}\left[\frac{kD_+}{\bar{z}} + \overline{\left(\frac{kD_-}{\bar{z}}\right)}\right] + o(z^{-1}), \quad u_2^h = \frac{i\Gamma}{2|k|}\left[\frac{kD_+}{\bar{z}} - \overline{\left(\frac{kD_-}{\bar{z}}\right)}\right] + o(z^{-1}) \quad \text{as} \quad z \to \infty \tag{3.24}$$

is imposed.

*Homogeneous solution to the inner problem.* It is now necessary to construct a homogeneous solution $\mathbf{u}^h$ that satisfies the boundary condition (3.24) at infinity and causes $\mathbf{u}^{(1)}$ [see (3.13)] to satisfy the boundary conditions (on the wake and the airfoil surface) deduced in appendix C. We begin by constructing a formal solution to this problem, which will subsequently be made uniformly valid. In order to do this, it is first necessary to consider the singularities at the leading and trailing edges. Thus, upon inserting (B 10) into the boundary condition (C 5)

† We have (in anticipation of the matching process) used the same symbol here as we did for the arbitrary constant in the inner solution (3.15). Thus (3.22) now effectively defines that constant.

we find that the term $-\beta d[x_1 \Delta^H u_1^{(0)}(x_1)]/dx_1$ causes $u_2^{(1)}(x_1)$ to behave like $(x_1+1)^{-\frac{3}{2}}$ at the leading edge ($x_1 = -1$) and like $(x_1-1)^{-\frac{1}{2}}$ at the trailing edge. (It is assumed, of course, that the camber $y_c(x_1)$ and thickness $b(x_1)$ go to zero fast enough at the leading and trailing edges to ensure that no other singularities occur in the boundary conditions (C 5) and (C 6).) These singularities can be removed by putting

$$\left.\begin{aligned} u_1^H &= u_1^h + \beta \frac{k_1}{|k|} S(k_1) \operatorname{Re}\left\{\frac{d}{dz} z\left[1-\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}\right]\right\}, \\ u_2^H &= u_2^h - \beta \frac{k_1}{|k|} S(k_1) \operatorname{Im}\left\{\frac{d}{dz} z\left[1-\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}\right]\right\}, \end{aligned}\right\} \tag{3.25}$$

where the branch cut is taken along the real axis from $-1$ to $+1$, $S(k_1)$ is the (complex conjugate†) Sears function [*see* (B 8)] and $\alpha\beta$ is the airfoil angle of attack. Since

$$\Delta u_1^H(x_1) = \Delta u_1^h(x_1), \quad \Delta u_2^H(x_1) = \Delta u_2^h(x_1) \quad \text{for} \quad |x_1| > 1, \tag{3.26}$$

$$\left.\begin{aligned} \Delta u_2^H(x_1) &= \Delta u_2^h(x_1) + 2\beta \frac{k_1}{|k|} S(k_1) \frac{d}{dx_1} x_1 \left(\frac{1-x_1}{1+x_1}\right)^{\frac{1}{2}} \\ \langle u_2^H(x_1)\rangle &= \langle u_2^h(x_1)\rangle \end{aligned}\right\} \quad \text{for} \quad -1 < x_1 < 1, \tag{3.27}$$

where $\langle u_2(x_1)\rangle$ denotes the average $\frac{1}{2}[u_2(x_1, +0) + u_2(x_1, -0)]$, and since

$$\mathbf{u}^H = \mathbf{u}^h + O(|z|^{-2}) \quad \text{as} \quad z \to \infty, \tag{3.28}$$

it is easy to see that, apart from eliminating the singular term in (C 5), $\mathbf{u}^H$ satisfies the same boundary conditions as $\mathbf{u}^h$ (including the condition (3.24) at infinity). More important, however, $u_1^H$ and $u_2^H$ are themselves functions of the form (3.14) wherein, in order to satisfy condition (3.24), we must put $f(x_2)$ equal to zero. Then, since $\Delta u_1^{(1)}(x_1) = \Delta u_2^{(1)}(x_1) = 0$ for $x_1 < -1$, it follows from (3.18), (3.24), (3.26) and (3.28) and the theory of piecewise analytic functions (Gakhov 1966, p. 25) that

$$u_1^H - iu_2^H = \frac{1}{2\pi i}\int_{-1}^{\infty} \frac{\Delta u_1^H(x_1) - i\Delta u_2^H(x_1)}{x_1 - z}\, dx_1, \tag{3.29a}$$

$$\overline{u_1^H} - i\overline{u_2^H} = \frac{1}{2\pi i}\int_{-1}^{\infty} \frac{\Delta \overline{u_1^H}(x_1) - i\,\Delta \overline{u_2^H}(x_1)}{x_1 - z}\, dx_1 \tag{3.29b}$$

for all $z$ outside the cut $-1 < x_1 < \infty$. By adding (or subtracting) the complex conjugate of the second equation to the first, it is possible to calculate $u_1^H$ (or $u_2^H$) everywhere outside the strip $-1 < x_1 < \infty$ once $\Delta u_1^H$ and $\Delta u_2^H$ are known. The required expressions for these quantities are given in appendix D [see (D 1)–(D 8)]. They relate $\Delta u_1^H$ and $\Delta u_2^H$ to the steady-flow solution, the geometry of the airfoil and an arbitrary constant $K_1$.

$K_1$ is determined by the requirement that the solution should satisfy the boundary condition (3.24) at infinity. Thus, by inserting (D 1) and (D 2) into

† Notice that we are using the time dependence $\exp(-ik_1 t)$ rather than the dependence $\exp(ik_1 t)$ used by Sears.

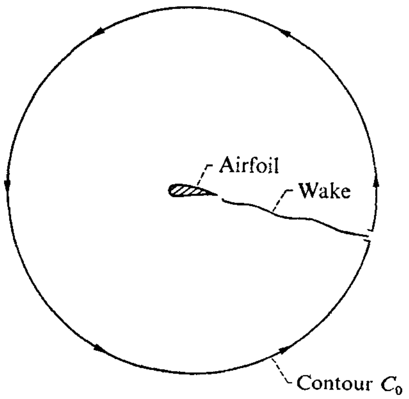


FIGURE 3. Contour for calculating circulation.

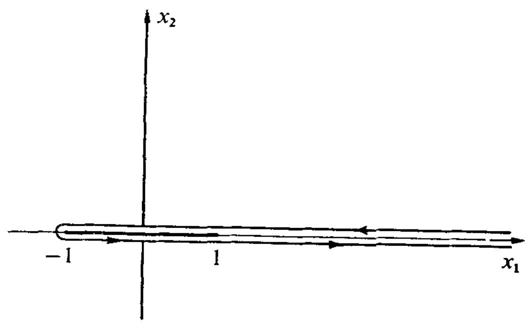


FIGURE 4. Deformed contour for calculating circulation.

(3.29) and integrating by parts, it can be shown that there exist constants $\tilde{a}$ and $\tilde{b}$ such that

$$u_1^H - iu_2^H \sim \tilde{a}/z, \quad \overline{u_1^H} - i\overline{u_2^H} \sim \tilde{b}/z \quad \text{as} \quad z \to \infty \quad \text{for} \quad \delta < \arg z < 2\pi - \delta \quad (3.30)$$

for any $\delta > 0$. Hence the solutions (3.29) are indeed compatible with the boundary condition (3.24). But it is easy to show that they will not satisfy this condition unless

$$\int_{C_0} \mathbf{u}^H . d\mathbf{S} = -\frac{i\pi\Gamma}{|k|}(\overline{kD_-} - kD_+), \tag{3.31}$$

where $C_0$ is the large circle shown in figure 3, which can be deformed into the contour shown in figure 4, so that

$$\int_{-1}^{\infty} \Delta u_1^H(x_1)\, dx_1 = \frac{i\pi\Gamma}{|k|}(\overline{kD_-} - kD_+). \tag{3.32}$$

This result is used in appendix D to show that $K_1$ is determined by (D 9).

The solution now satisfies all the equations and boundary conditions governing the problem. Nevertheless, it is non-uniformly valid at the leading and trailing edges because the second term on the right side of (3.25) causes the $O(\alpha\epsilon)$ term to be more singular at these points than the $O(\epsilon)$ (i.e. Sears) solution. This difficulty can be overcome by using the method of matched asymptotic expansions. The 'inner solution' for a flat-plate airfoil is constructed in appendix E. However, this result shows that the zeroth-order outer solution already possesses appropriate singular behaviour at the leading edge and therefore that the non-uniformity arises only because the singularity is not located at the right place. Thus, even though the problem is elliptic, the method of strained co-ordinates (Van Dyke 1964, p. 100) can be used instead of the more complicated method of matched asymptotic expansions. In fact, we use a modification of the usual procedure suggested by Pritulo (1962; see Van Dyke 1964, pp. 72 and 73). Thus, substituting (3.15), (3.17) and (B 14) into expansion (3.2), introducing the 'slightly strained' co-ordinate $\eta \equiv \xi_1 + i\xi_2 \equiv z/(1-i\alpha\beta)$ into the result, and re-expanding for small $\alpha\beta\eta$ yields†

$$u_n = \exp(-ik_1 t)\,(u_n^{(0)}(\xi_1,\xi_2) + \alpha u_n^p(\xi_1,\xi_2) + \alpha u_n^H(\xi_1,\xi_2) + \alpha\beta D_\xi\, {}^b u_n^{(0)}(\xi_1,\xi_2)$$
$$-\alpha\beta\frac{k_1}{|k|}S(k_1)\,\mathrm{Re}\left\{i^{(n-1)}\left[1-\left(\frac{\eta-1}{\eta+1}\right)^{\frac{1}{2}}\right]\right\}) + O(\alpha^2) \quad \text{for} \quad n = 1, 2, \quad (3.33)$$

where
$$D_\xi \equiv \xi_2\,\partial/\partial\xi_1 - \xi_1\,\partial/\partial\xi_2, \quad (3.34)$$

and the $O(\alpha)$ terms are now no more singular than $\mathbf{u}^{(0)}(\xi_1,\xi_2)$ at $\eta = \pm 1$ (or any other point). The quantities $u_1$ and $u_2$ still denote components of $\mathbf{u}$ along the $x_1$ and $x_2$ co-ordinate axes. They are related to the components $u_1'$ and $u_2'$ along the $\xi_1$ and $\xi_2$ axes (figure 2) by

$$\left.\begin{aligned} u_1'(\xi_1,\xi_2) &= u_1(\xi_1,\xi_2) - \alpha\beta\exp(-ik_1 t)\,u_2^{(0)}(\xi_1,\xi_2) + O(\alpha^2),\\ u_2'(\xi_1,\xi_2) &= u_2(\xi_1,\xi_2) + \alpha\beta\exp(-ik_1 t)\,u_1^{(0)}(\xi_1,\xi_2) + O(\alpha^2).\end{aligned}\right\} \quad (3.35)$$

### 3.2. *Discussion of solution*

The solution to the problem is now complete. The velocity field in the outer region can be calculated from (3.24), while the velocity in the inner region is determined by (3.33) with $\mathbf{u}^{(0)}$ and ${}^b\mathbf{u}^{(0)}$ given in appendix B, $\mathbf{u}^p$ given by (3.15) and $\mathbf{u}^H$ given by (3.29) and (D 1)–(D 10). Of course, we cannot evaluate the integrals in these formulae until the geometry of the airfoil and the steady-state potential flow solution are specified. This will eventually be done for a flat-plate airfoil at an angle of attack to the mean flow.

Equations (3.15) and (3.16) show that the $O(\alpha\epsilon)$ particular solution is proportional to $k$ as $k \to \infty$, while the results of appendix $B$ show that the $O(\epsilon)$ particular solution is proportional to $k^0$. Hence the expansion in $\alpha$ is actually an expansion in powers of $\alpha k$ and as such is certainly not uniformly valid in frequency space. However, this behaviour does indicate that the steady flow will have its greatest influence on the fluctuating lift at higher reduced frequencies. (Of course compressibility effects will invalidate the entire solution when $k$ becomes too large.)

As long as the thickness $b(x_1)$ and mean camber line $y_c(x_1)$ (figure 2) vanish at a reasonable rate when $x_1$ approaches $\pm 1$, the present result will be uniformly valid in all regions of the $x_1$, $x_2$ plane. But in order to achieve this uniformity, we have had to strain the solutions at the leading and trailing edges. An important consequence of this straining is a change in the apparent orientation of the airfoil. A similar effect occurs in steady-airfoil theory. But in that case a uniformly valid expansion can be obtained simply by solving the problem in the proper (airfoil aligned) co-ordinate system. In the present case this procedure would lead to a divergent integral for the part of the velocity induced by the wake.

We have already indicated that the steady-state velocity field influences the wavelength of the incident vorticity wave while leaving its amplitude unchanged. But the particular solution shows that the amplitude and direction as well as the wavelength of the associated velocity field are altered by the steady flow. Indeed,

† Of course, we understand that $u_1^{(0)}(\xi_1, \xi_2)$ and $u_1^{(0)}(x_1, x_2)$ are identical functions of their respective arguments; i.e. if $u_1^{(0)}(x_1, x_2) = ax_1 + x_2^2$, then $u_2^{(1)}(\xi_1, \xi_2) = a\xi_1 + \xi_2^2$.

(3.23) shows that at large distances from the airfoil the change in amplitude of the unsteady vortical velocity field is proportional to the steady circulation about the airfoil. This result is a reflexion of the fact that the steady lift produces the slowest decaying part of the potential flow about the airfoil (although the numerical calculations show that most of the distortion occurs in the vicinity of the airfoil).

It follows from (2.3) and (3.23) that the far-field gust velocity is

$$\left\{\frac{-\epsilon U k_2}{|k|}\exp\{i[\mathbf{k}.\mathbf{x}-\alpha\,\mathrm{Re}\{k(W^{(1)}-W_0)\}-k_1 t]\},\right.$$
$$\left.\frac{\epsilon U k_1}{|k|}\exp\{i[\mathbf{k}.\mathbf{x}-\alpha\,\mathrm{Re}\{k(W^{(1)}-W_0)\}-k_1 t]\}\right\}, \quad (3.36)$$

which in the case of a lifting airfoil differs from the gust

$$\left\{\frac{-\epsilon U k_2}{|k|}\exp i\{i(\mathbf{k}.\mathbf{x}-k_1 t)\},\quad \frac{\epsilon U k_1}{|k|}\exp\{i(\mathbf{k}.\mathbf{x}-k_1 t)\}\right\} \quad (3.37)$$

that is imposed in the strictly linear (Sears) problem. In the present case the perturbation potential $W^{(1)}(z)$ behaves like $\ln|\mathbf{x}|$ as $\mathbf{x}\to\infty$, and hence its contribution to the exponent in (3.36) cannot be neglected no matter how small $\alpha$ may be. Far from the airfoil, where the 'scattered' part of the unsteady velocity goes to zero and only the gust remains, the latter quantity must itself satisfy equations of continuity and momentum that are linearized about the steady potential flow. However, this flow disturbs the region at infinity enough that the solutions of these equations are of the type (3.36) rather than of the type (3.37). The gust (3.36) differs from the gust (3.37) in that the former is frozen relative to an observer moving along the steady-state potential flow streamlines with a speed $U$, while the latter is frozen with respect to an observer moving along the real axis with this speed.

The components of the amplitude $\mathbf{A}\equiv\{-k_2\epsilon U/|k|,\ k_1\epsilon U/|k|\}$ of these gusts are not independent because the associated velocity field satisfies the continuity equation only when the transverse wave condition $\mathbf{A}.\mathbf{k}=0$ holds.

## 4. Fluctuating lift

### 4.1. *General formulae*

In most applications it is necessary to know the net fluctuating lift caused by the gust. In order to determine this quantity, we first calculate the fluctuating pressure force $p^{\pm}_{\mathrm{surf}}$ on the upper/lower surface. If we introduce the expansion

$$p' = \exp(-ik_1 t)\,[p^{(0)}(\boldsymbol{\xi})+\alpha\tilde{p}^{(1)}(\boldsymbol{\xi})+\ldots] \quad (4.1)$$

of the pressure fluctuation in the $\boldsymbol{\xi}=\{\xi_1,\xi_2\}$ co-ordinate system, $p^{\pm}_{\mathrm{surf}}$ can be written as [(2.4) and figure 2]

$$p^{\pm}_{\mathrm{surf}} = \epsilon\exp(-ik_1 t)\left\{p^{(0)}(\xi_1,\pm 0)+\alpha\left[\tilde{p}^{(1)}(\xi_1,\pm 0)+\left(y_c\pm\frac{b}{2}\right)\left(\frac{\partial p^{(0)}}{\partial\xi_2}\right)_{\xi_2=\pm 0}\right]+O(\alpha^2)\right\}. \quad (4.2)$$

After (2.3), (2.4), (3.1), (3.33), (3.35) and (4.1) have been inserted into the momentum equation (2.2) and (C 3), (B 1), (B 2) and (B 14) have been used to simplify the results, equating the $\xi_2$ component of the $O(\epsilon)$ terms yields

$$(\partial\tilde{p}^{(0)}/\partial\xi_2)_{\xi_2=\pm 0} = 0 \quad \text{for} \quad |\xi_1| < 1, \tag{4.3}$$

while equating the $\xi_1$ component of the $O(\alpha\epsilon)$ terms yields

$$\frac{d}{d\xi_1}\left[\Delta\tilde{p}^{(1)}(\xi_1) - \frac{k_2}{|k|}\exp(ik_1\xi_1)\,\Delta v_1^{(1)}(\xi_1) + \langle v_1^{(1)}(\xi_1)\rangle\,\Delta^H u_1^{(0)}(\xi_1)\right] - \frac{k_2^2}{|k|}\exp(ik_1\xi_1)\,\Delta v_2^{(1)}(\xi_1)$$
$$= \left(ik_1 - \frac{\partial}{\partial\xi_1}\right)[\Delta u_1^p(\xi_1) + \Delta u_1^H(\xi_1)] \quad \text{for} \quad |\xi_1| < 1. \tag{4.4}$$

Hence $L'$, the net fluctuating lift per unit span acting on the airfoil, can be written in the form

$$L'/\tfrac{1}{2}\rho c U^2\epsilon = (L_0'/\tfrac{1}{2}\rho c U^2\epsilon) + \alpha(L_1'/\tfrac{1}{2}\rho c U^2\alpha\epsilon), \tag{4.5}$$

where
$$(2L_0'/c\rho U^2\epsilon)\exp(ik_1 t) = -\int_{-1}^{1}\Delta p^{(0)}(\xi_1)\,d\xi_1$$

is the usual linearized response function (i.e. the Sears function), and the $O(\epsilon\alpha)$ contribution to the lift is given by

$$\frac{L_1'}{\tfrac{1}{2}\rho c U^2\alpha\epsilon} = -\exp(-ik_1 t)\int_{-1}^{1}\Delta\tilde{p}^{(1)}(\xi_1)\,d\xi_1. \tag{4.6}$$

In order to evaluate this integral, we first note that the steady-state circulation around the airfoil is just equal to

$$\Gamma \equiv \frac{1}{2\pi}\int_{-1}^{1}\Delta v_1^{(1)}(x_1)\,dx_1, \tag{4.7}$$

that the condition that $\mathbf{v}.\hat{\mathbf{n}}$ be zero on the surface of the airfoil implies that $\Delta v_2^{(1)} = db/dx_1$, and that the imposition of the Kutta condition at the trailing edge (for both steady and unsteady flows) implies that $\Delta v_1^{(1)}(1) = \Delta p^{(1)}(1) = 0$. Then, using (4.4) to eliminate $\Delta\tilde{p}^{(1)}$ in (4.6), integrating by parts and inserting (3.15), (3.16) and (3.12), and integrating by parts again and using (3.13), (3.19), (D 1), (B 4) and the first part of (D 10) to simplify the results yields

$$\frac{L_1'}{\tfrac{1}{2}c\rho U^2\alpha\epsilon} = \exp(-ik_1 t)\left\{\int_{-1}^{1}[1 + ik_1(x_1 - 1)]\,\Delta u_1^H(x_1)\,dx_1 - (\gamma^+ D_+ - \overline{\gamma^- D_-})\right\}$$
$$+\exp(-ik_1 t)\left\{\int_{-1}^{1}\langle v_1^{(1)}(x_1)\rangle\,\Delta^H u_1^{(0)}(x_1)\,dx_1\right.$$
$$\left. -\frac{k_1^3}{|k|}\int_{-1}^{1}\exp(ik_1x_1)(1 + x_1)\,b(x_1)\,dx_1 + \frac{ik_1}{2|k|}(kD_+ + \overline{kD_-})\int_{-1}^{1}b(x_1)\,dx_1\right\}, \tag{4.8}$$

where
$$\gamma^\pm = \frac{1}{2|k|}\int_{-1}^{1}[(x_1 - 1)\,kk_1 \pm i\bar{k}]\,\Delta v_1^{(1)}(x_1)\,dx_1. \tag{4.9}$$

The fluctuating lift can now be calculated by substituting (D 8) into (4.8) and carrying out the integration. This cannot be done explicitly without first introducing a specific formula for the steady flow, but it is possible to reduce the multiple integrals to a single quadrature by interchanging the order of integration and integrating by parts. While the resulting formulae are quite complicated in the general case, they simplify enormously when the airfoil thickness is put equal to zero. Unfortunately, because of the coupling that results from the distortion of the incident gust by the steady-state potential flow, it is not possible, as it is in the case of linearized steady flow, to superpose the effects of thickness, camber and angle of attack. However, airfoil thickness probably has only an unimportant influence on the unsteady lift and will not be considered further.

### 4.2. *Airfoil with zero thickness*

In order to obtain a specific formula for the fluctuating lift on a zero-thickness airfoil, we first substitute (D 8) into (4.8) and use (F 1) and (F 2) to simplify the results. We then interchange the order of integration, evaluate the inner integrals [with the aid of (F 5)], and use (D 10) [in which one of the integrations can be performed by virtue of (F 2)] to eliminate $K_1$. Finally, after some additional rearrangement [with the aid of (F 3) and (F 4)], we find that

$$\frac{L_1'}{\frac{1}{2}c\rho U^2\alpha\epsilon} = 2\exp(-ik_1t)\left[-\frac{\pi i\Gamma k_1}{|k|}(D_+-\overline{D_-}) + ik_1\int_{-1}^{1}(1-x_1)R_0(x_1)\,dx_1 - C(k_1)\int_{-1}^{1}R_0(x_1)\,dx_1\right], \quad (4.10)$$

where

$$C(k_1) \equiv H_1^{(1)}(k_1)/[H_1^{(1)}(k_1)-iH_0^{(1)}(k_1)] \quad (4.11)$$

is the (complex conjugate) Theodorsen function (Theodorsen 1935),

$$R_0 = \frac{i}{|k|}\left(\frac{1+x_1}{1-x_1}\right)^{\frac{1}{2}}\left\{k_1\exp(ik_1x_1)\,\mathrm{Re}\,ka_0 + \frac{d}{dx_1}[q_0^+(x_1)-\overline{q_0^-(x_1)}]\right\}, \quad (4.12)$$

$$q_0^\pm(x_1) = \frac{k}{2}\exp\left(\frac{\pm ikx_1}{2}\right)\left[\int_{\mp\infty}^{x_1}\langle v_2^{(1)}(x_1)\rangle\exp\left(\pm\frac{i\bar{k}x_1}{2}\right)dx_1 + D_\pm\int_{\mp\infty}^{x_1}\langle v_2^{(1)}(x_1)\rangle\exp\left(\mp\frac{ikx_1}{2}\right)dx_1\right] \quad (4.13)$$

and

$$a_0 \equiv \langle W^{(1)}(x_0)\rangle - W_0. \quad (4.14)$$

$a_0$ is determined both by the value $\langle W^{(1)}(x_0)\rangle$ of $\langle W^{(1)}(x_1)\rangle$ at the essentially arbitrary point $x_1 \equiv x_0$, where the surface of the airfoil crosses the $x_1$ axis, and by the difference between the arbitrary constant used to set the level of the imaginary part of $W^{(1)}$ and the arbitrary constant $e_0$.

In most cases it is probably necessary to evaluate the integrals in (4.10)–(4.13) numerically. Fortunately, there are a large number of interesting shapes for which they can be expressed in terms of known functions. In fact, for a flat-plate airfoil at an angle of attack to the flow, they can be expressed in terms of the combinations

$$J_\pm(z) \equiv J_0(z) \pm iJ_1(z), \quad H_\pm(z) \equiv H_1^{(1)}(z) \mp iH_0^{(1)}(z) \quad (4.15)$$

of Bessel and Hankel functions. Since this configuration is completely characterized by its angle of attack, we can suppose that the expansion parameter $\alpha$ is equal to this quantity and put $\beta = 1$ (figure 2). Then the complex-conjugate steady-flow velocity perturbation $\zeta^{(1)}$ is given by (Jones & Cohen 1957)

$$\zeta^{(1)} \equiv v_1^{(1)} - iv_2^{(1)} = i\left[1 - \left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}\right], \tag{4.16}$$

where the branch cut of the square root is taken from $-1$ to $+1$ along the real axis. Inserting this into (4.10) by way of (4.12) and (4.13), carrying out the integrations, and rearranging with the aid of (B 9) yields

$$\frac{L_1'}{c\rho U(\epsilon U)\pi} = \frac{\alpha \exp(-ik_1 t)}{|k|}\left\{k_1\left[-\left(i\,\mathrm{Re}\,ka_0 + \frac{4k_1k_2}{|k|^2}\right)S(k_1) + \Theta_+\left(\frac{k}{2}\right) - \overline{\Theta_-\left(\frac{k}{2}\right)}\right] + iC(k_1)\left[\Lambda_+\left(\frac{k}{2}\right) - \overline{\Lambda_-\left(\frac{k}{2}\right)}\right]\right\}, \tag{4.17}$$

where we have put

$$\left.\begin{aligned}\Lambda_\pm(z) &\equiv i\pi z^2\,\mathrm{Re}\,\{H_\pm(z)\,\overline{J_\pm(z)}\},\\ \Theta_\pm(z) &\equiv i\,\frac{zJ_1(z)\,\pi\,\mathrm{Re}\,\{\overline{J_\pm(z)}\,H_\pm(z)\} \mp \overline{J_\pm(z)}}{J_\pm(z)}.\end{aligned}\right\} \tag{4.18}$$

This solution is valid only for positive values of $k_2$. By modifying the analysis given above, it is possible to show that the relation $L_1'(k_1, -k_2) = -L_1'(k_1, k_2)$ can be used to extend it to negative values. However, it is much easier to establish this result from symmetry arguments.

### 4.3. *Discussion of flat-plate results*

Equation (4.8) can be used to determine the unsteady force acting on an airfoil of any shape, but the calculation will usually involve quadratures. We have succeeded, however, in expressing the fluctuating lift for a flat-plate airfoil entirely in terms of Bessel functions. The result is given by (4.17).

It follows from (4.14) that the first term in this equation,

$$-ik_1\,|k|^{-1}\exp(-ik_1 t)\,\mathrm{Re}\,\{k[\langle W^{(1)}(x_0)\rangle - W_0]\}, \tag{4.19}$$

is simply a correction to the linear (Sears) solution,

$$\frac{L_0'}{c\rho U(\epsilon U)\pi} = \frac{k_1}{|k|}\exp(-ik_1 t)\,S(k_1), \tag{4.20}$$

for the constant phase factor introduced into the gust (3.36) by the arbitrary choice of level of the steady-state complex potential function $W^{(1)}$ relative to the constant $e_0 = \mathrm{Im}\,W_0$ and by the choice of the precise (i.e. correct to $O(\alpha)$) vertical location of the airfoil. In fact, it is not hard to show [by integrating (4.17)] that $\langle W^{(1)}(x_0)\rangle - W_0 = i(x_0 + e - e_0)$, where $e$ is the constant that was introduced in expansion (3.20) of the steady-state potential and $x_0$ is the value of $x_1$ where the airfoil crosses the real axis. Hence this term vanishes when $e_0$ is put equal to $e + x_0$, and in what follows we shall always assume this has been done. Then (3.20) and (3.22) imply that

$$W^{(1)} - W_0 = i\ln|\mathbf{x}| - ix_0 + O(|\mathbf{x}|^{-1}) \quad \text{as } x_1 \to \infty \text{ with } x_2 \text{ finite}. \tag{4.21}$$

Since $\alpha x_0$ is the height $x_c$ of the centre of the airfoil above the real axis, the exponents in (3.36) become

$$i[k_1(x_1-t)+k_2(x_2-x_c)+\alpha k_2 \ln|\mathbf{x}|+O(|\mathbf{x}|^{-1})] \quad \text{as } x_1 \to -\infty \text{ with } x_2 \text{ finite.} \tag{4.22}$$

Thus neglecting the first term in (4.17) amounts to nothing more than referencing the phase of the gust to the vertical position of the centre of the airfoil.

Since elimination of the secular behaviour at the leading and trailing edges has caused the $O(\epsilon)$ pressure distribution to be rotated into the plane of the airfoil, there will be an unsteady drag force of order $\alpha\epsilon$. This result should be compared with those of Glauert (1929) and Jones (1957) for the unsteady motion of a thin airfoil, in which the drag is $O(\epsilon^2)$.

An $O(\alpha\epsilon)$ correction to the Sears formula was obtained by Horlock (1968), who adopted a more-or-less heuristic approach to the problem. His result can be shown to consist of a correction to the Sears function due to the orientation of the gust relative to the airfoil plus a term (in the present notation)

$$-\alpha k_2|k|^{-1}\exp(-ik_1t)[J_0(k_1)-iJ_1(k_1)], \tag{4.23}$$

which arises from the inertia contribution $v_1^{(1)}(x_1, \pm 0)\,u_1^{(0)}(x_1, \pm 0)$ to the pressure force in (4.4). But our analysis shows that this term is exactly cancelled by one that enters the formula for the lift through the particular solution $\mathbf{u}^p$ and can therefore be attributed to the distortion of the gust by the steady-state potential flow field (an effect not accounted for by Horlock). Thus (4.17) differs considerably from the results obtained by Horlock. We should emphasize again that the present analysis is a systematic ('exact') theory that accounts for all $O(\alpha\epsilon)$ terms, including those associated with the distortion of the gust by the steady-state potential flow field of the airfoil. It shows that this gust distortion effect has a strong influence on the behaviour of the response function.

If $k_2$ is allowed to approach zero while $k_1$ is held fixed, so that only the upwash component of the gust velocity remains, $L_1'$ will vanish and the fluctuating lift will be completely determined by the Sears function. Horlock's expression for the lift also vanishes when the chordwise gust velocity goes to zero. But unlike his result, (4.17) depends on both the axial and the transverse wavenumber $k_1$ and $k_2$ and not just on the axial wavenumber $k_1$. Thus the present analysis not only exhibits the effects of gust distortion on the response function, but also shows for the first time how this function is influenced by the wavenumber in the direction perpendicular to the plane of the airfoil.

We first consider the low frequency limit, wherein $k_1$ and $k_2$ both go to zero. Since $\Lambda_\pm(z) \to 0$ and $\Theta_\pm(z) \to \mp i$ as $z \to 0$ and $S(k_1) \to C(k_1) \to 1$ as $k_1 \to 0$, it follows that

$$L_1'/c\rho U(\epsilon U)\pi \sim -4\alpha k_1^2 k_2/|k|^3 \quad \text{as} \quad k_1, k_2 \to 0. \tag{4.24}$$

This at first glance appears to be a surprising result since in the quasi-steady approximation the fluctuating lift is given by

$$L_{\text{q.s.}}'/\epsilon\rho c U\pi = u_2 + 2\alpha u_1, \tag{4.25}$$

so that if we simply take $u_1$ and $u_2$ to be the disturbance velocities $-k_2/|k|$ and $k_1/|k|$ at infinity, we find that the $O(\alpha)$ contribution should be $-2k_2/|k|$. However, (3.15) and (3.16) show that the average upwash velocity $\langle u_2^p(x_1)\rangle$ induced at a finite point $x_1$ of the real axis by the $O(\alpha)$ contribution to the particular solution is

$$\frac{i}{|k|}[q^+(x_1)+\overline{q^-(x_1)}]-\frac{ik_1}{|k|}\exp(ik_1x_1)\,\mathrm{Re}\left\{k\left[\frac{W^{(1)}(x+i0)+W^{(1)}(x-i0)}{2}\right]\right\}, \tag{4.26}$$

where $q^{\pm}$ are given by (D 7). Upon evaluating the integrals for the case of a flat-plate airfoil, we find that the term $i[q^+(x_1)+\overline{q^-(x_1)}]/|k|$ does not vanish in the limit as $k_1$ and $k_2$ approach zero but behaves like

$$\frac{i}{2|k|}\left(\frac{k^2}{\bar{k}}-\frac{\bar{k}^2}{k}\right)-\frac{i}{|k|}\left(\frac{kD_+-\overline{kD_-}}{2}\right). \tag{4.27}$$

The second term of this expression arises from the portion of the particular solution that does not match the incoming gust at infinity (and is cancelled out by the requirement that the homogeneous solution satisfy the boundary condition (3.24)). Hence only the first term, which can be written as

$$2k_2/|k|-4k_1^2k_2/|k|^3, \tag{4.28}$$

is actually associated with oncoming gust. Thus the net upwash velocity that this gust induces at the airfoil is not given by $k_1/|k|$ but by

$$u_2=\frac{k_1}{|k|}+\alpha\left(\frac{2k_2}{|k|}-4\frac{k_1^2k_2}{|k|^3}\right). \tag{4.29}$$

When this is inserted (together with $u_1=-k_2/|k|+O(\alpha)$) into (4.25), we do indeed recover (4.24). This result is a consequence of the non-uniform limit

$$\lim_{|\mathbf{x}|\to\infty}\ \lim_{|k|\to 0}\ u_2 \neq \lim_{|k|\to 0}\ \lim_{|\mathbf{x}|\to\infty} u_2. \tag{4.30}$$

Physically, it implies that the steady-state potential flow field decays so slowly that the gust arriving at the airfoil surface always suffers a certain limiting amount of distortion in its upwash velocity no matter how long its wavelength is.

Now consider the high frequency limit, wherein $k\to\infty$, with $k_2>0$. Then, since it follows from the asymptotic behaviour of the Bessel functions that

$$\left.\begin{aligned} H_{\pm}(z) &\sim -i\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\exp\left[i\left(z-\frac{\pi}{4}\right)\right]\left[(1\pm 1)+\frac{1}{8z}(3\mp 1)\right]\\ J_{\pm}(z) &\sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\exp\left[\pm i\left(z-\frac{\pi}{4}\right)\right]\end{aligned}\right\}\quad \text{as}\quad z\to\infty, \tag{4.31}$$

we find that

$$\mathrm{Re}\{\overline{J_{\pm}(z)}\,H_{\pm}(z)\}=O(z^{-2})\quad\text{as}\quad z\to\infty, \tag{4.32}$$

and as a result that

$$\Theta_{\pm}\sim(k/\bar{k})^{\frac{1}{2}}\exp(\mp ik_1). \tag{4.33}$$

Hence
$$\frac{L_1'}{c\rho U(\epsilon U)\pi}=2i\alpha\frac{k_1k_2}{|k|^2}\exp(ik_1)+O(k^{-1})\quad\text{as}\quad k\to\infty. \tag{4.34}$$

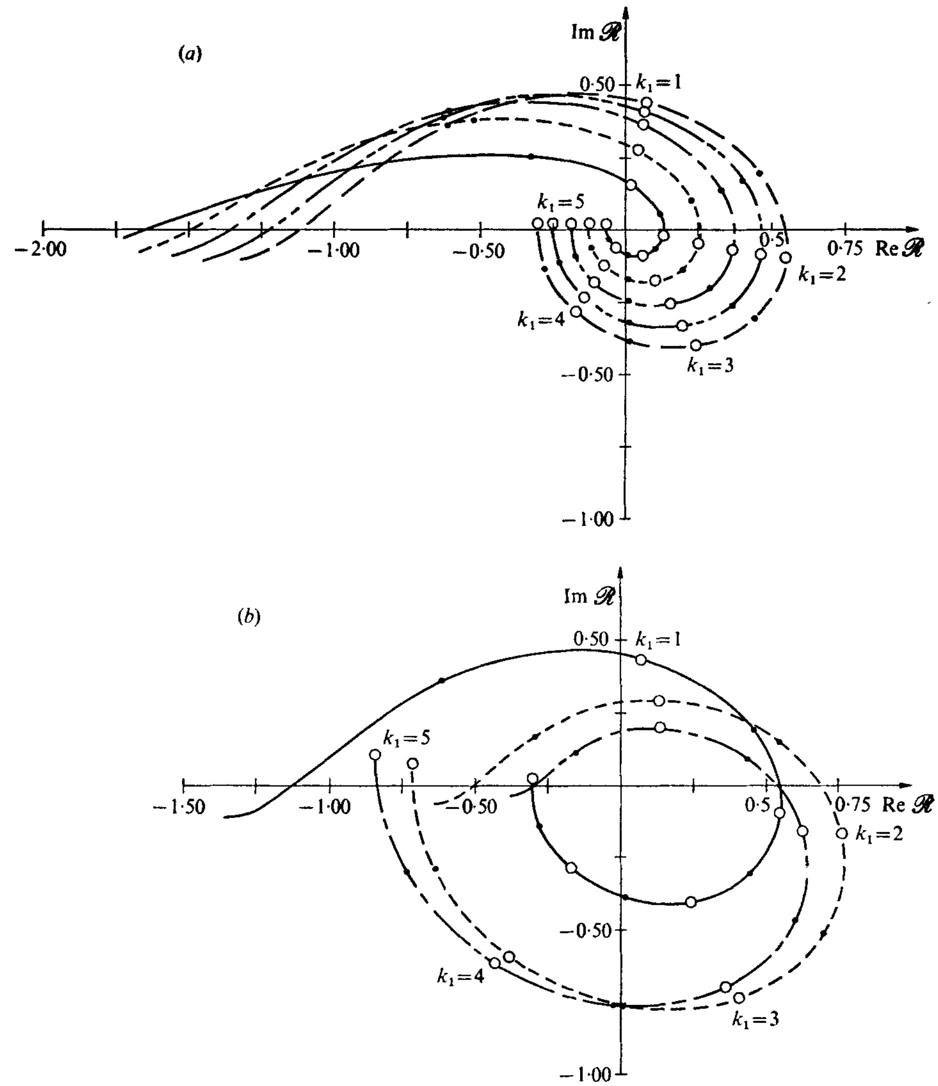


FIGURE 5. Response function for flat-plate airfoil. (a) $0 < k_2 \leqslant 1$. ——, $k_2 = 0{\cdot}2$; – – –, $k_2 = 0{\cdot}4$; —·—, $k_2 = 0{\cdot}6$; —··—, $k_2 = 0{\cdot}8$; ——, $k_2 = 1{\cdot}0$. (b) $1 \leqslant k_2 \leqslant 5$. ——, $k_2 = 1{\cdot}0$; – – –, $k_2 = 3{\cdot}0$; —·—, $k_2 = 5{\cdot}0$.

Thus, if $k_1 \to \infty$ with $k_2$ held fixed or if $k_2 \to \infty$ with $k_1$ held fixed, the lift fluctuation will have a $k^{-1}$ decay rate, which is faster than the $k_1^{-\frac{1}{2}}$ decay of the $O(\epsilon)$ (Sears) lift fluctuation $L_0'$. Hence the effects of the steady flow on the lift become less important.

However, $L_1'$ does not decay at all when $k_1$ and $k_2$ are allowed to approach infinity at the same rate, so that the steady-state potential flow has its greatest effect on the fluctuating lift at higher frequencies. Of course, $L_1'$ will eventually become larger than the $O(\epsilon)$ contribution $L_0'$ (no matter how small $\alpha$ is), and the expansion will be invalid.

The dimensionless response function

$$\mathscr{R} \equiv L_1' \exp(ik_1 t)/[\pi c \rho U(\epsilon U)\alpha] \tag{4.35}$$

[calculated from (4.17)] is plotted in figure 5. The curves simply show that this function varies smoothly between the various limits that were discussed above.

The authors would like to thank Professor W. R. Sears for his encouragement during the course of this work.

## Appendix A

Here we obtain asymptotic expansions of $\mathscr{K}_\pm(z)$ as $z \to \infty$. Suppose that $z$ is large. Then, owing to the analyticity of its integrand, it is always possible to choose the path of integration in (3.2) such that $\zeta^{(1)}(z)$ can be replaced by its asymptotic value $i\Gamma/z$. Consequently

$$\mathscr{K}_\pm(z) \sim i\Gamma \int_{\mp\infty}^{z} \frac{\exp(\pm\frac{1}{2}i\bar{k}z)}{z}\,dz \quad \text{as} \quad z \to \infty. \tag{A 1}$$

These quantities differ from exponential integrals only in the location of the branch cuts and their asymptotic expansions can be found by using the procedures developed for these integrals. Thus, for example, by combining the method used in § 3.2, p. 32 ,of Lebedev (1965) with that used in exercise 6, p. 41, of that reference, it is easy to show that

$$\left.\begin{aligned} \mathscr{K}_+(z) &\sim (2\Gamma/\bar{k}z)\exp(\tfrac{1}{2}i\bar{k}z) \quad \text{for} \quad 0 \leqslant \arg z \leqslant 2\pi, \\ \mathscr{K}_-(z) &\sim (-2\Gamma/\bar{k}z)\exp(-\tfrac{1}{2}i\bar{k}z) \quad \text{for} \quad |\arg z| \leqslant \pi. \end{aligned}\right\} \tag{A 2}$$

## Appendix B

Here we list and in some cases develop further certain properties of the $O(\epsilon)$ (linear) solution that are needed for the present analysis. This (Sears 1941) solution is the superposition

$$\mathbf{u}^{(0)} = -(\hat{\imath}k_2 - \hat{\jmath}k_1)\,|k|^{-1} e^{i\mathbf{k}\cdot\mathbf{x}} + {}^{H}\mathbf{u}^{(0)} \tag{B 1}$$

of the linear gust (3.37) and a homogeneous solution ${}^{H}\mathbf{u}^{(0)}$ that decays like $z^{-2}$ (since it has zero circulation) as $z \to \infty$. The components of ${}^{H}\mathbf{u}^{(0)}$ satisfy the Cauchy–Riemann equations

$$\frac{\partial\, {}^{H}u_1^{(0)}}{\partial x_1} + \frac{\partial\, {}^{H}u_2^{(0)}}{\partial x_2} = 0, \quad \frac{\partial\, {}^{H}u_1^{(0)}}{\partial x_2} - \frac{\partial\, {}^{H}u_2^{(0)}}{\partial x_1} = 0, \tag{B 2}$$

with the $x_1$ component having the odd symmetry

$${}^{H}u_1^{(0)}(x_1, +0) = -\,{}^{H}u_1^{(0)}(x_1, -0) \tag{B 3}$$

along the real axis. The jump in this component across the wake is given by

$$\Delta\, {}^{H}u_1^{(0)} = \frac{2k_1}{|k|}\,\Omega_0 \exp[ik_1(x_1 - 1)] \quad \text{for} \quad 1 < x_1 < \infty, \tag{B 4}$$

where

$$\frac{2k_1}{|k|}\,\Omega_0 \equiv \Delta\, {}^{H}u_1^{(0)}(1) = \frac{4k_1}{|k|}\exp(ik_1)\left[\frac{J_0(k_1) + iJ_1(k_1)}{H_1^{(1)}(k_1) - iH_0^{(1)}(k_1)}\right], \tag{B 5}$$

and the $J$'s and $H$'s denote the usual Bessel and Hankel functions. Its jump across the airfoil is related to the corresponding pressure jump by

$$\left(-ik_1+\frac{d}{dx_1}\right)\Delta^H u_1^{(0)}(x_1) = -\frac{d}{dx_1}\Delta p^{(0)}(x_1) \tag{B 6}$$

and satisfies the zero-circulation condition

$$ik_1\int_{-1}^{1}\Delta^H u_1^{(0)}(x_1)\,dx_1 = 2k_1\,\Omega_0/|k|. \tag{B 7}$$

Since $\Delta p^{(0)}$ is related to the complex-conjugate Sears function

$$S(k_1) \equiv \{\tfrac{1}{2}i\pi k_1[H_1^{(1)}(k_1)-iH_0^{(1)}(k_1)]\}^{-1} \tag{B 8}$$

by

$$\Delta p^{(0)} = -\frac{2k_1}{|k|}S(k_1)\left(\frac{1-x_1}{1+x_2}\right)^{\frac{1}{2}}, \tag{B 9}$$

(B 6) can be integrated with the aid of (B 7) to show with the aid of (B 3) that

$${}^H u_1^{(0)}(x_1,\pm 0) = \pm\tfrac{1}{2}[h_s(x_1)+h_b(x_1)] \quad \text{for} \quad -1 < x_1 < 1, \tag{B 10}$$

where

$$h_s(x_1) \equiv \frac{2k_1}{|k|}S(k_1)\left(\frac{1-x_1}{1+x_1}\right)^{\frac{1}{2}} \tag{B 11}$$

has a square-root singularity at $x_1 = -1$ and

$$h_b(x_1) \equiv \frac{2k_1}{|k|}\Big[\Omega_0\exp[ik_1(x_1-1)]+ik_1S(k_1)\exp(ik_1x_1) \times\int_1^{x_1}\exp(-ik_1x_1)\left(\frac{1-x_1}{1+x_1}\right)^{\frac{1}{2}}dx_1\Big] \tag{B 12}$$

remains bounded at both ends. In fact, (B 12) takes on the values

$$b_b(-1) = 0, \quad h_b(1) = 2k_1|k|^{-1}\Omega_0 \tag{B 13}$$

at these two points. Hence it follows that $\mathbf{u}^{(0)}$ can be written as

$$\mathbf{u}^{(0)} = {}^b\mathbf{u}^{(0)}+\frac{k_1}{|k|}S(k_1)\left[\hat{\mathbf{i}}\,\mathrm{Re}\left\{-i\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}\right\}-\hat{\mathbf{j}}\,\mathrm{Im}\left\{-i\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}\right\}\right], \tag{B 14}$$

where ${}^b\mathbf{u}^{(0)}$ is bounded and satisfies the Cauchy–Riemann equations (B 2).

## Appendix C

Here we develop the linearized boundary conditions which hold on the airfoil and across the wake. Our method for identifying the airfoil and wake surfaces is shown in figure 2. The location of the latter is unknown at this stage of the development, but to the order of approximation of the analysis it can be characterized by a function whose general form is indicated in figure 2. Then, as is well known from the theory of unsteady inviscid flows, the boundary condition (2.16) is equivalent to

$$u_1 d(\alpha f^{\pm})/dx_1 = u_2 \quad \text{for} \quad x_2 = f^{\pm}, \tag{C 1}$$

while the velocities $\mathrm{V}^{\pm}$ just above and below the wake must satisfy

$$\left(\frac{\partial}{\partial t}+V_1^{\pm}\frac{\partial}{\partial x_1}\right)(\alpha g+\epsilon\tilde{g}+\alpha\epsilon\tilde{\tilde{g}}) = V_2^{\pm}. \tag{C 2}$$

As is usual in thin-airfoil theory, we 'transfer' the boundary conditions onto the real ($x_1$) axis by assuming that the various quantities can be expanded in a Taylor series about $x_2 = 0$. Performing this expansion in (C 1), inserting (3.2), equating to zero the coefficients of like powers of $\alpha$ and using (B 1) and (B 2) (to eliminate $u_2^{(0)}$) yields

$$u_2^{(0)}(x_1, \pm 0) = 0 \quad \text{for} \quad -1 < x_1 < 1 \tag{C 3}$$

and
$$u_2^{(1)}(x_1, \pm 0) = d[f_\pm(x_1)\, u_1^{(0)}(x_1, \pm 0)]/dx_1 \quad \text{for} \quad -1 < x_1 < 1. \tag{C 4}$$

The $f$'s can be expressed in terms of the mean camber $\alpha y_c(x_1)$, the angle of attack $\alpha\beta$ and the thickness $\alpha b(x_1)$ by the relation $f_\pm(x_1) = y_c(x_1) - \beta x_1 \pm \frac{1}{2}b(x_1)$, which, together with (B 1) and (B 3), allows us to replace (C 4) by

$$\Delta u_2^{(1)}(x_1) = \frac{d}{dx_1}\left\{[y_c(x_1) - \beta x_1]\,\Delta^H u_1^{(0)}(x_1) - \frac{k_2}{|k|}\, b(x_1)\exp(ik_1 x_1)\right\} \quad \text{for} \quad |x_1| < 1, \tag{C 5}$$

$$\langle u_2^{(1)}(x_1)\rangle = \frac{d}{dx_1}\left\{\frac{k_2}{|k|}\,[\beta x_1 - y_c(x_1)]\exp(ik_1 x_1) + \frac{b(x_1)}{4}\,\Delta^H u_1^{(0)}(x_1)\right\} \quad \text{for} \quad |x_1| < 1. \tag{C 6}$$

Carrying out the Taylor series expansions in each term of (C 2), inserting (3.1), (3.2), (B 1) and (B 2), subtracting the results and equating to zero the coefficients of like powers of $\alpha$ and $\epsilon$ yields

$$\Delta u_2^{(0)}(x_1) = 0 \quad \text{for} \quad 1 < x_1 < \infty \tag{C 7}$$

and

$$\Delta u_2^{(1)}(x_1) = -\frac{2k_1}{|k|}\,\Omega_0 \frac{d}{dx_1}\{\Psi^{(1)}(x_1, 0)\exp[ik_1(x_1 - 1)]\} \quad \text{for} \quad 1 < x_1 < \infty, \tag{C 8}$$

where we have used the result that $g(x_1) = -\Psi^{(1)}(x_1, 0)$ (along with the convention that the zero steady-state flow streamline coincides with the stagnation streamline).

It is also necessary to ensure that the pressure is continuous across the wake. To this end we insert the expansions (2.3), (2.4), (3.1) and (3.2) into the momentum equation (2.2) and equate to zero the coefficients of like powers of $\alpha$ and $\epsilon$ to obtain

$$\frac{1}{\alpha}\frac{\partial p_s}{\partial x_2} = -\frac{\partial}{\partial x_1} v_2^{(1)}, \quad \frac{\partial p^{(0)}}{\partial x_2} = \left(ik_1 - \frac{\partial}{\partial x_1}\right) u_2^{(0)}. \tag{C 9}$$

Then, after expanding the pressure on either side of the wake about $x_2 = 0$ and inserting (2.4) and (3.2), we find [in view of (C 7)] that the pressure will be continuous across the wake only if $\Delta p^{(0)}(x_1) = \Delta p^{(1)}(x_1) = 0$ for $1 < x_1 < \infty$. Using these relations together with the expansions (2.3), (2.4), (3.1) and (3.2) in the momentum equation (2.2) now yields

$$\left(-ik_1 + \frac{d}{dx_1}\right)\Delta u_1^{(1)}(x_1) = -\frac{d}{dx_1}[v_1^{(1)}(x_1, 0)\,\Delta u_1^{(0)}(x_1)], \tag{C 10}$$

since (B 1) and (B 2) now show that $\Delta(\partial u_1^{(0)}/\partial x_2)_{x_2=0} = 0$ across the wake. Upon solving this first-order differential equation, using (B 1) and (B 4), integrating the results by parts and using the fact that $v_1^{(1)} \equiv \partial\Phi^{(1)}/\partial x_1$, we find that†

$$\Delta u_1^{(1)}(x_1) = -\frac{2k_1\Omega_0}{|k|}\frac{d}{dx_1}\{[\Phi^{(1)}(x_1,0)+K_0]\exp[ik_1(x_1-1)]\} \quad \text{for} \quad 1 < x_1 < \infty, \tag{C 11}$$

where $K_0$ is an arbitrary constant of integration.

## Appendix D

Here we use some of the techniques of unsteady thin-airfoil theory to deduce expressions for $\Delta u_1^H$ and $\Delta u_2^H$ across the airfoil and its wake. It follows from inserting (3.12), (3.13), (3.15), (3.16), (3.19) and (3.25)–(3.27) into the boundary conditions (C 5), (C 8) and (C 11) and then using (B 1) and (B 10) that

$$\Delta u_1^H(x_1) = -\frac{2k_1}{|k|}\Omega_0\frac{d}{dx_1}\{[\Phi^{(1)}(x_1,0)+K_1]\exp[ik_1(x_1-1)]\} \quad \text{for} \quad 1 \leqslant x_1 < \infty, \tag{D 1}$$

$$\Delta u_2^H(x_1) = -\frac{2k_1}{|k|}\Omega_0\frac{d}{dx_1}\{\Psi^{(1)}(x_1,0)\exp[ik_1(x_1-1)]\}+\frac{ik_1}{|k|}\exp(ik_1x_1)\,\mathrm{Re}\{k\,\Delta W^{(1)}(1)\} \quad \text{for} \quad 1 < x_1 < \infty \tag{D 2}$$

and

$$\begin{aligned}\Delta u_2^H(x_1) = {} & -\frac{i}{|k|}[r_+(x_1)+r_-(x_1)]+\frac{ik_1}{|k|}\exp(ik_1x_1)\,\mathrm{Re}\{k\,\Delta W^{(1)}(x_1)\} \\ & -\frac{k_2}{|k|}\frac{d}{dx_1}[b(x_1)\exp(ik_1x_1)]+\frac{d}{dx_1}[y_c(x_1)\,\Delta^H u_1^{(0)}(x_1)] \\ & -\beta\frac{\mathrm{d}}{dx_1}[x_1h_b(x_1)] \quad \text{for} \quad -1 < x_1 < 1,\end{aligned} \tag{D 3}$$

where

$$r_\pm(x_1) \equiv \frac{k^2}{4}\exp(\pm\tfrac{1}{2}ikx_1)\left[\int_{-1}^{x_1}\Delta\zeta^{(1)}(x_1)\exp(\pm\tfrac{1}{2}i\bar{k}x_1)\,dx_1 - D_\pm\int_{-1}^{x_1}\overline{\Delta\zeta^{(1)}(x_1)}\exp(\mp\tfrac{1}{2}ikx_1)\,dx_1\right], \tag{D 4}$$

$h_b(x_1)$ is given by (B 12) and $K_0$ has been replaced by the new arbitrary constant $K_1$. Hence the homogeneous solutions $u_1^H$ and $u_2^H$ can be calculated once the jump $\Delta u_1^H(x_1)$ in the range $-1 < x_1 < 1$ and the constant $K_1$ are known. In order to determine the former, we use the Plemelj formulae (Gakhov 1966, p. 25) to take the limiting values of (3.29) as $z$ approaches the real axis, subtract the complex conjugate of the result obtained from (3.29*b*) from that obtained from (3.29*a*), and then use the boundary condition (C 6) [with (3.13) and (3.15)–(3.17) inserted] to eliminate $u_2^H(x_1)$ from the ensuing expression. This yields

$$R(x_1) = \frac{1}{2\pi}\mathrm{P}\int_{-1}^{1}\frac{\Delta u_1^H(\tilde{x}_1)}{\tilde{x}_1-x_1}d\tilde{x}_1-\frac{k_1\Omega_0}{\pi|k|}\int_1^\infty\frac{d\{[\Phi^{(1)}(\tilde{x}_1,0)+K_1]\exp[ik_1(\tilde{x}_1-1)]\}/d\tilde{x}_1}{\tilde{x}_1-x_1} \times d\tilde{x}_1 \quad \text{for} \quad -1 < x_1 < 1, \tag{D 5}$$

† In the steady-flow solution $\Phi^{(1)}$ is discontinuous across the wake, but we do not bother here to distinguish between $\Phi^{(1)}(x_1+0, 0)$ and $\Phi^{(1)}(x_1-0, 0)$ since they differ only by a constant that can always be absorbed into the arbitrary constant $K_0$.

where

$$R(x_1) \equiv -\frac{k_2}{|k|}\frac{d}{dx_1}\{[y_c(x_1)-\beta x_1]\exp(ik_1x_1)\}+\frac{1}{4}\frac{d}{dx_1}[b(x_1)\,\Delta^H u_1^{(0)}(x_1)]$$
$$-\frac{i}{|k|}[q^+(x_1)+\overline{q^-(x_1)}]+\frac{ik_1}{|k|}\exp(ik_1x_1)\,\mathrm{Re}\{k[\langle W^{(1)}(x_1)\rangle - W_0]\}, \quad \text{(D 6)}$$

with

$$q^{\pm}(x_1) \equiv \frac{k^2}{4}\exp(\pm\tfrac{1}{2}ikx_1)\left[\int_{\mp\infty}^{x_1}\langle\zeta^{(1)}(x_1)\rangle\exp(\pm\tfrac{1}{2}i\bar{k}x_1)\,dx_1\right.$$
$$\left.-D_{\pm}\int_{\mp\infty}^{x_1}\langle\zeta^{(1)}(x_1)\rangle\exp(\mp\tfrac{1}{2}ikx_1)\,dx_1\right]. \quad \text{(D 7)}$$

Equation (D 5) possesses a whole family of solutions (Gakhov 1966, p. 428). However, the Kutta condition requires that the velocity jump $\Delta u_1^H(\tilde{x}_1)$ remain finite at the trailing edge. The only solution with this property is

$$\Delta u_1^H(x_1) = -\frac{2}{\pi}\left(\frac{1-x_1}{1+x_1}\right)^{\frac{1}{2}}\left[\mathrm{P}\int_{-1}^{1}\left(\frac{1+\tilde{x}_1}{1-\tilde{x}_1}\right)^{\frac{1}{2}}\frac{R(\tilde{x}_1)}{\tilde{x}_1-x_1}\,d\tilde{x}_1\right]+\frac{k_1\Omega_0}{|k|}\int_1^{\infty}\left(\frac{\tilde{x}_1+1}{\tilde{x}_1-1}\right)^{\frac{1}{2}}$$
$$\times\left(\frac{d\{[\Phi^{(1)}(\tilde{x}_1,0)+K_1]\exp[ik_1(\tilde{x}_1-1)]\}/d\tilde{x}_1}{\tilde{x}_1-x_1}\right)d\tilde{x}_1 \quad \text{for } -1<x_1<1. \quad \text{(D 8)}$$

In order to determine the constant $K_1$, we first insert (D 1) into (3.32). In performing the indicated integration, we follow the procedure used in linear theory and assume that $k_1$ has a small positive imaginary part that we can put equal to zero after the integrations are carried out. Then

$$\int_{-1}^{1}\Delta u_1^H(x_1)\,dx_1 = -\frac{2k_1\Omega_0}{|k|}[\Phi^{(1)}(1,0)+K_1]-\frac{i\pi\Gamma}{|k|}(kD_+-\overline{kD_-}). \quad \text{(D 9)}$$

Hence, upon integrating both sides of (D 8) and interchanging the order of integration, we find that $K_1$ is determined by

$$-\frac{1}{2}\int_{-1}^{1}\Delta u_1^H(x_1)\,dx_1 = \frac{k_1\Omega_0}{|k|}[\Phi^{(1)}(1,0)+K_1]+\frac{i\pi\Gamma}{|k|}(kD_+-\overline{kD_-})$$
$$=\int_{-1}^{1}\left(\frac{1+x_1}{1-x_1}\right)^{\frac{1}{2}}R(x_1)\,dx_1+\frac{k_1\Omega_0}{|k|}\int_1^{\infty}\left[\left(\frac{x_1+1}{x_1-1}\right)^{\frac{1}{2}}-1\right]\frac{d}{dx_1}$$
$$\times\{[\Phi^{(1)}(x_1,0)+K_1]\exp[ik_1(x_1-1)]\}\,dx_1. \quad \text{(D 10)}$$

## Appendix E

Here we investigate the unsteady flow in the vicinity of the leading edge of a flat plate. The (non-linearized) steady-flow velocity $\zeta^{\mathrm{I}}$ about a flat plate of unit length at an angle of attack $\alpha_0$ to an oncoming stream is

$$\zeta^{\mathrm{I}} = |\mathbf{U}_0|\left[\cos\alpha_0 - i\left(\frac{\eta-1}{\eta+1}\right)^{\frac{1}{2}}\sin\alpha_0\right], \quad \text{(E 1)}$$

where $\mathbf{U}_0$ is the free-stream velocity, the circulation is adjusted to satisfy the Kutta condition at the trailing edge, and $\eta = \xi_1 + i\xi_2$ denotes a co-ordinate system

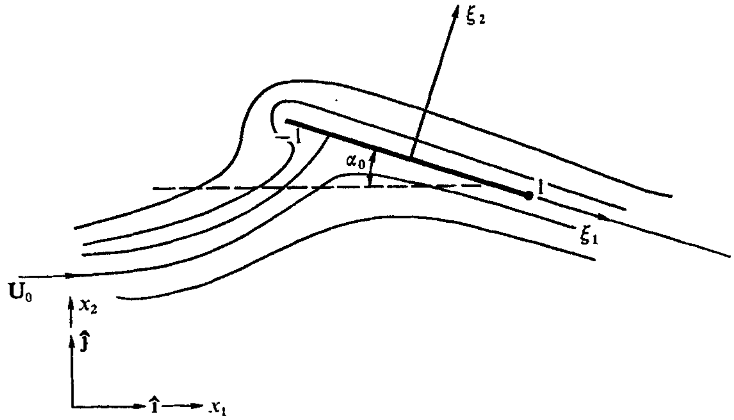


FIGURE 6. Co-ordinate system for flat-plate airfoil.

aligned with the plate as shown in figure 6. In the vicinity of the leading edge (i.e. for $\eta$ near $-1$), this becomes

$$\zeta^{\mathrm{I}} \sim |\mathbf{U}_0| \left(\frac{2}{\eta+1}\right)^{\frac{1}{2}} \sin \alpha_0. \tag{E 2}$$

Now suppose that the flow in this region is unsteady. We cannot, in general, linearize the velocity about the mean flow; but we can neglect its derivatives with respect to time in comparison with spatial derivatives. (That is, we can treat the flow as quasi-steady in this region.) Thus the velocity will be given by (E 2) with $\alpha_0$ and $\mathbf{U}_0$ taken to be the effective instantaneous angle of attack and free-stream velocity. As a result, there exist constants $\mathbf{a}^{(0)}$ and $\mathbf{a}^{(1)}$ (which depend on $k_1$ and $k_2$ and which can be determined by matching with the outer solution) such that

$$\mathbf{U}_0 = \hat{\mathbf{\imath}} + \epsilon \exp(-ik_1 t)\,[\mathbf{a}^{(0)} + \alpha \mathbf{a}^{(1)}] + o(\alpha\epsilon) \tag{E 3}$$

and

$$\alpha_0 = \alpha + \epsilon \exp(-ik_1 t)\,[a_2^{(0)} + \alpha a_2^{(1)}] + o(\alpha\epsilon). \tag{E 4}$$

Hence, when terms that are clearly of higher order in $\alpha$ and $\epsilon$ are neglected, (E2) becomes

$$\zeta^{\mathrm{I}} \sim \left(\frac{2}{\eta+1}\right)^{\frac{1}{2}} (\alpha + \epsilon \exp(-ik_1 t)\,\{a_2^{(0)} + \alpha[a_2^{(1)} + a_1^{(0)}]\}). \tag{E 5}$$

The first term $\alpha 2^{\frac{1}{2}}/(\eta+1)^{\frac{1}{2}}$ is just the linearized steady-flow solution. The constant $a_2^{(0)}$ can clearly be adjusted such that the second term

$$\epsilon \exp(-ik_1 t)\, a_2^{(0)} 2^{\frac{1}{2}}/(\eta+1)^{\frac{1}{2}}$$

will match with the dominant term of the zeroth-order solution to within an error $O(\alpha\epsilon)$ (since $\eta = z + O(\alpha)$). The last term is the $O(\alpha\epsilon)$ correction for the unsteady solution.

## Appendix F

Here we list certain properties of the linearized approximation to the steady-flow velocity field around a zero-thickness airfoil. The axial velocity possesses the odd symmetry

$$v_1^{(1)}(x_1, +0) = -v_1^{(1)}(x_1, -0) \quad \text{for} \quad -\infty < x_1 < \infty \tag{F 1}$$

across the real axis, so that $v_1^{(1)}(x_1)$ must vanish ahead of and behind the airfoil (where $\Delta v_1^{(1)}(x_1) = 0$). Hence $\Phi^{(1)}(x_1, 0)$ must be constant along the wake, so that

$$\Phi^{(1)}(x_1, 0) = \Phi^{(1)}(1, 0) \quad \text{for} \quad 1 < x_1 < \infty. \tag{F 2}$$

On the surface of the airfoil,

$$\langle W^{(1)}(x_1)\rangle = \tilde{a}_0 + i\Psi^{(1)} = \tilde{a}_0 - i\int_{x_0}^{x_1} \langle v_2^{(1)}(x_1)\rangle\, dx_1 \quad \text{for} \quad |x_1| < 1, \tag{F 3}$$

where $\tilde{a}_0 \equiv \langle W^{(1)}(x_0)\rangle$ is the value of $\langle W^{(1)}(x_1)\rangle$ at the point where the airfoil crosses the real axis. Moreover, the average upwash velocity is related to the shape of the airfoil by

$$\langle v_2^{(1)}(x_1)\rangle = y_c'(x_1) - \beta \quad \text{for} \quad -1 < x_1 < 1, \tag{F 4}$$

while the tangential velocity is related to the average upwash velocity by

$$\Delta v_1^{(1)}(x_1) = -\frac{2}{\pi}\left(\frac{1-x_1}{1+x_1}\right)^{\frac{1}{2}} \int_{-1}^{1} \left(\frac{1+\tilde{x}_1}{1-\tilde{x}_1}\right)^{\frac{1}{2}} \frac{\langle v_2^{(1)}(\tilde{x}_1)\rangle}{\tilde{x}_1 - x_1}\, d\tilde{x}_1. \tag{F 5}$$

## REFERENCES


GAKHOV, F. D. 1966 *Boundary Value Problems*. Pergamon.

GLAUERT, H. 1929 The force and moment on an oscillating aerofoil. *Aero. Res. Counc. R. & M.* no. 1242.

HORLOCK, J. H. 1968 Fluctuating lift forces on airfoils moving through transverse and chordwise gusts. *J. Basic Engng, Trans. A.S.M.E.* **90**, 494.

JONES, D. S. 1957 The unsteady motion of a thin aerofoil in an incompressible fluid. *Comm. Pure Appl. Math.* **10**, 1.

JONES, R. & COHEN, D. 1957 Aerodynamics of wings at high speeds. In *High Speed Aerodynamics and Jet Propulsion*, vol. VII. *Aerodynamic Components of Aircraft at High Speeds*, p. 3. Princeton University Press.

KÜSSNER, H. G. 1940 Das zweidimensionale Problem der beliebig bewegten Tragfläche unter Berücksichtigung von Partialbewegungen der Flüssigkeit. *Luftfahrtorsch.* **17**, 355.

LEBEDEV, N. N. 1965 *Special Functions and Their Applications*. Prentice-Hall.

LIGHTHILL, M. J. 1951 A new approach to thin aerofoil theory. *Aero. Quart.* **2**, 193.

MILNE-THOMSON, L. M. 1962 *Theoretical Hydrodynamics*, 4th ed. Macmillan.

NAUMANN, H. & YEH, H. 1972 Lift and pressure fluctuations of a cambered airfoil under periodic gusts and application to turbomachinery. *A.S.M.E. Paper*, no. 72-GT-30.

PRITULO, M. F. 1962 On the determination of uniformly accurate solutions of differential equations by the method of perturbation coordinates. *J. Appl. Math. Mech.* **26**, 661.

SEARS, W. R. 1941 Some aspects of non-stationary airfoil theory and its practical application. *J. Aero. Sci.* **8**, 104.

THEODORSEN, T. 1935 General theory of aerodynamic instability and mechanism of flutter. *N.A.C.A. Tech. Rep.* no. 496.

VAN DYKE, M. 1954 Supersonic flow past oscillating airfoils including non-linear thickness effects. *N.A.C.A. Tech. Rep.* no. 1183.

VAN DYKE, M. 1964 *Perturbation Methods in Fluid Mechanics*. Academic.